THE LORENTZ-INVARIANT CLASSICAL WAVE EQUATION FOR A MASSLESS SCALAR FIELD

A Four-Dimensional Formulation in Minkowski Spacetime

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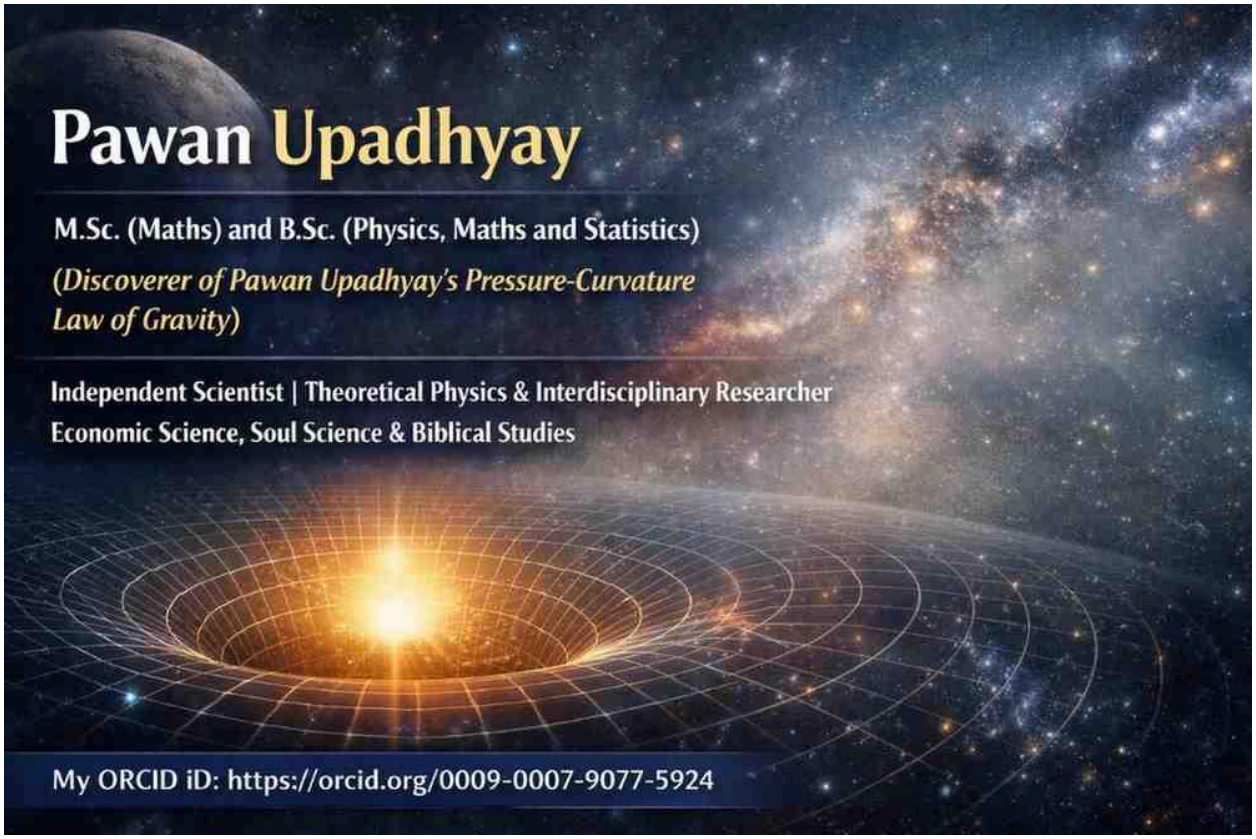

ABSTRACT

The classical wave equation is traditionally written as

$\partial^2\Phi/\partial t^2 = c^2 \nabla^2 \Phi$

and is commonly associated with the propagation of disturbances through a physical medium.

However, the mathematical structure of the wave equation is more general than mechanical wave motion. When formulated in four-dimensional spacetime using the coordinates (x, y, z, ct), the equation takes the Lorentz-invariant form

$$\Box\Phi = 0$$

This equation describes the propagation of a free massless scalar field in Minkowski spacetime.

This paper examines the mathematical transition from the conventional three-dimensional wave equation to its four-dimensional relativistic form. The spacetime metric, d'Alembertian operator, plane-wave solutions, dispersion relation, null propagation, and massless character of the field are developed. The distinction between mechanical waves and classical field waves is also examined. Finally, the scalar wave equation is compared with the electromagnetic wave equations derived from Maxwell's theory.

1. INTRODUCTION

The wave equation occupies a central position in classical mathematical physics. It appears in the theory of vibrating strings, acoustics, elasticity, electromagnetism, and many other areas of physics.

In its familiar form, the scalar wave equation is

$$\partial^2\Phi/\partial t^2 = c^2\nabla^2\Phi$$

where Φ represents the quantity or field undergoing wave propagation and c represents the characteristic propagation velocity.

In mechanical systems, Φ may represent a displacement, pressure variation, or another physical quantity associated with matter. Such examples naturally lead to the idea that a wave is a disturbance of a material medium.

The development of relativistic physics provides a broader perspective. A field can possess wave dynamics without representing the mechanical displacement of a material substance. In particular, the electromagnetic field propagates through vacuum and is described classically by Maxwell's equations.

The purpose of this paper is to investigate the four-dimensional formulation of the classical wave equation using the spacetime coordinates

(x, y, z, ct).

The central equation considered is

$$\Box\Phi = 0$$

where $\Box$ denotes the d'Alembertian operator.

The objective is not to replace the ordinary wave equation but to demonstrate that its relativistic structure is naturally expressed through a four-dimensional spacetime operator.

2. THE CLASSICAL WAVE EQUATION

Let

$\Phi = \Phi(x, y, z, t)$

be a scalar field.

The three-dimensional Laplacian is

$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$

The classical wave equation is therefore

$\partial^2\Phi/\partial t^2 = c^2\nabla^2\Phi$

Expanding the Laplacian,

$\partial^2\Phi/\partial t^2 = c^2(\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2)$

Dividing by c^2 and rearranging,

$\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2 - (1/c^2)\partial^2\Phi/\partial t^2 = 0$

This equation already contains the mathematical relationship between spatial variation and temporal variation that characterizes wave propagation.

3. FOUR-DIMENSIONAL SPACETIME COORDINATES

We introduce the four coordinates

$x^\mu = (x, y, z, ct)$

The field becomes

$\Phi = \Phi(x, y, z, ct)$

The use of ct rather than t is dimensionally convenient because x, y, z, and ct all have dimensions of length.

The fourth coordinate is nevertheless fundamentally different from the three spatial coordinates. It represents time multiplied by the invariant speed c.

The spacetime interval is

$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$

Equivalently,

$ds^2 = dx^2 + dy^2 + dz^2 - d(ct)^2$

The minus sign distinguishes Minkowski spacetime from ordinary Euclidean four-dimensional space.

4. MINKOWSKI METRIC

Using the metric convention

$g_{\mu\nu} = \text{diag}(1, 1, 1, -1)$

the spacetime interval can be written as

$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

The inverse metric has the same diagonal form:

$g^{\mu\nu} = \text{diag}(1, 1, 1, -1)$

This metric determines how spatial and temporal derivatives combine to produce a Lorentz-invariant differential operator.

The essential feature is that the temporal component enters with the opposite sign from the spatial components.

5. THE FOUR-DIMENSIONAL WAVE OPERATOR

Define the four-dimensional derivative

$\partial_\mu = \partial/\partial x^\mu$

The Lorentz-invariant wave operator is defined by

$\Box = g^{\mu\nu}\partial_\mu\partial_\nu$

For the metric chosen above,

$\Box = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 - \partial^2/\partial(ct)^2$

Since

$\partial/\partial(ct) = (1/c)\partial/\partial t$

we obtain

$\partial^2/\partial(ct)^2 = (1/c^2)\partial^2/\partial t^2$

Therefore,

$\Box = \nabla^2 - (1/c^2)\partial^2/\partial t^2$

6. THE LORENTZ-INVARIANT CLASSICAL WAVE EQUATION

The source-free equation is

$\Box\Phi = 0$

Expanding,

$\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2 - \partial^2\Phi/\partial(ct)^2 = 0$

Equivalently,

$\nabla^2\Phi - (1/c^2)\partial^2\Phi/\partial t^2 = 0$

Rearranging,

$\partial^2\Phi/\partial t^2 = c^2\nabla^2\Phi$

Thus, the conventional classical wave equation is recovered exactly.

The four-dimensional formulation therefore does not introduce a different wave equation. Instead, it reveals the spacetime structure of the equation.

7. LORENTZ INVARIANCE

A Lorentz transformation connects the coordinates of two inertial observers:

$x^\mu \rightarrow x'^\mu$

The defining property of the transformation is the invariance of the spacetime interval:

$ds'^2 = ds^2$

For a scalar field,

$\Phi'(x') = \Phi(x)$

The d'Alembertian is constructed from the Minkowski metric and therefore has invariant form.

Consequently,

$\Box\Phi = 0$

has the same mathematical form in all inertial frames.

This property distinguishes the relativistic wave equation from a generic wave equation written in an arbitrary coordinate system.

8. PLANE-WAVE SOLUTION

Consider the plane-wave solution

$$\Phi = A\, e^{i(k\cdot r - \omega t)}$$

where

$$r = (x, y, z)$$

and

$$k = (k_x, k_y, k_z)$$

For this field,

$$\nabla^2\Phi = -k^2\Phi$$

and

$$\partial^2\Phi/\partial t^2 = -\omega^2\Phi$$

Substitution into the wave equation gives

$$-k^2\Phi + (\omega^2/c^2)\Phi = 0$$

For a nonzero field,

$$\omega^2/c^2 = k^2$$

Therefore,

$$\omega^2 = c^2k^2$$

and hence

$$\omega = ck$$

for positive frequency.

This is the dispersion relation for the massless wave.

9. THE FOUR-WAVE-VECTOR

Define the wave four-vector by

$k^\mu = (k_x, k_y, k_z, \omega/c)$

The plane wave can be expressed in four-dimensional notation as

$\Phi = A\, e^{ik_\mu x^\mu}$

The wave equation requires

$k_\mu k^\mu = 0$

Using the adopted metric,

$k_\mu k^\mu = k^2 - \omega^2/c^2$

Therefore,

$k_\mu k^\mu = 0$

implies

$\omega^2 = c^2 k^2$

The four-wave-vector is consequently a null vector.

10. MASSLESS CHARACTER

The relativistic scalar-field equation with mass m is

$(\Box + m^2c^2/\hbar^2)\Phi = 0$

When

$m = 0$

the mass term disappears and the equation becomes

$\Box\Phi = 0$

Thus, the source-free scalar wave equation is the massless limit of the relativistic scalar-field equation.

The corresponding dispersion relation is

$\omega^2 = c^2 k^2$

The absence of a mass-dependent term is characteristic of the massless case.

11. NULL PROPAGATION AND THE LIGHT CONE

The spacetime interval is

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$$

For a null trajectory,

$$ds^2 = 0$$

Therefore,

$$dx^2 + dy^2 + dz^2 = c^2dt^2$$

Defining

$$d\ell^2 = dx^2 + dy^2 + dz^2$$

we obtain

$$d\ell = c\,dt$$

Consequently,

$$d\ell/dt = c$$

The wave equation is therefore associated with propagation along the null structure of Minkowski spacetime.

The set of null directions forms the light cone.

12. CLASSICAL FIELD VERSUS MECHANICAL WAVE

A central conceptual issue is the distinction between a classical wave and a mechanical wave.

A mechanical wave requires a physical system whose mechanical degrees of freedom participate in the disturbance.

For example, a string displacement u may satisfy

$$\partial^2u/\partial t^2 = v^2\partial^2u/\partial x^2$$

Here u describes a mechanical displacement.

The scalar field Φ in

$$\Box\Phi = 0$$

does not necessarily have such an interpretation.

It can instead represent a fundamental classical field.

Therefore, the mathematical existence of a wave equation does not by itself imply the existence of a material medium.

13. ELECTROMAGNETIC WAVES

Electromagnetic radiation is described fundamentally by Maxwell's equations.

In vacuum, Maxwell's equations imply wave equations for the electric and magnetic fields:

□E = 0

and

□B = 0

These equations have the same d'Alembertian structure as

□Φ = 0

However, the electromagnetic field is not a scalar field. It possesses vector and tensor structure.

The electromagnetic field can be represented covariantly by the antisymmetric field tensor F_μv.

Therefore, the scalar equation should not be identified as the complete electromagnetic theory.

Rather, it demonstrates the general mathematical structure shared by massless relativistic waves.

14. LAGRANGIAN FORMULATION

The equation

□Φ = 0

can be obtained from a Lorentz-invariant classical Lagrangian.

For a free real scalar field, consider

ℒ = ½∂_μΦ ∂^μΦ

The action is

S = ∫ℒ d⁴x

Applying the Euler-Lagrange field equation produces

$\partial_\mu \partial^\mu \Phi = 0$

Therefore,

$\Box \Phi = 0$

This result demonstrates that the wave equation can be formulated consistently within classical relativistic field theory.

The field does not have to be interpreted mechanically for the variational formulation to exist.

15. ENERGY AND MOMENTUM

A classical scalar field can possess energy and momentum.

For the free scalar field, the energy-momentum tensor can be written, with the corresponding sign convention, as

$T^{\mu\nu} = \partial^\mu \Phi \, \partial^\nu \Phi - \tfrac{1}{2} g^{\mu\nu} \partial_\alpha \Phi \, \partial^\alpha \Phi$

The field equation implies the conservation relation

$\partial_\mu T^{\mu\nu} = 0$

Thus, the field can transport conserved physical quantities even though it is not interpreted as a mechanical displacement of matter.

This provides an important distinction between the concepts of field propagation and material motion.

16. SOURCE-FREE AND SOURCE-DRIVEN EQUATIONS

The present paper considers the source-free equation

$\Box \Phi = 0$

A source can be introduced through an equation of the form

$\Box \Phi = J$

Here J represents a source term, with its precise normalization depending on the physical theory and convention.

The source-free equation describes free propagation, whereas the sourced equation describes a field influenced by an external or internal source.

This distinction is analogous to the difference between free electromagnetic radiation and electromagnetic fields generated by charges and currents.

17. WHY FOUR DIMENSIONS MATTER

The expression

$\Phi = \Phi(x, y, z, ct)$

makes explicit that the field is defined over spacetime.

The wave equation then becomes

$\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2 - \partial^2\Phi/\partial(ct)^2 = 0$

The equation cannot be understood correctly by treating all four coordinates as equivalent Euclidean directions.

The temporal coordinate has a different metric sign.

Thus, the essential structure is not simply four coordinates but rather four-dimensional Lorentzian spacetime.

This Lorentzian structure produces the causal light-cone geometry characteristic of relativistic wave propagation.

18. DISCUSSION

The four-dimensional formulation provides a deeper interpretation of the classical wave equation.

The ordinary form

$\partial^2\Phi/\partial t^2 = c^2\nabla^2\Phi$

emphasizes the relationship between spatial curvature and temporal acceleration.

The relativistic form

$\Box\Phi = 0$

emphasizes that the field exists on spacetime and that its differential equation is compatible with Lorentz symmetry.

The two descriptions are mathematically equivalent when the Minkowski spacetime structure is understood.

The four-dimensional form is therefore not merely a change of notation. It exposes the geometric principle underlying relativistic wave propagation.

19. LIMITATIONS

The equation

$$\Box\Phi = 0$$

should not be interpreted as a universal equation for all waves.

Different physical systems require different dynamical equations.

Mechanical waves depend on properties of matter such as density, elasticity, and pressure.

Electromagnetic waves require Maxwell's equations.

The scalar equation is therefore best interpreted specifically as the equation governing a free massless scalar field.

Its conceptual importance lies in demonstrating how wave propagation can be expressed independently of a mechanical medium.

20. CONCLUSION

The classical wave equation

$$\partial^2\Phi/\partial t^2 = c^2\nabla^2\Phi$$

can be naturally expressed in four-dimensional spacetime using the coordinates

$$x^\mu = (x, y, z, ct)$$

With the Minkowski metric

$$g_{\mu\nu} = \mathrm{diag}(1, 1, 1, -1)$$

the corresponding wave operator is

$$\Box = g^{\mu\nu}\partial_\mu\partial_\nu$$

and explicitly,

$$\Box = \nabla^2 - (1/c^2)\partial^2/\partial t^2$$

The source-free equation is therefore

$$\Box\Phi = 0$$

A plane-wave solution,

$\Phi = A\, e^{i(k\cdot r - \omega t)}$

leads to the dispersion relation

$\omega^2 = c^2k^2$

In four-vector language, this is expressed as

$k_\mu k^\mu = 0$

showing that the wave four-vector is null.

The analysis establishes that the classical massless scalar wave equation is naturally a Lorentz-invariant equation on Minkowski spacetime. Its mathematical structure does not require a mechanical medium. Instead, it describes the propagation of a classical field whose physical interpretation must be supplied by the relevant field theory.

FINAL MATHEMATICAL SUMMARY

Spacetime coordinates:

$x^\mu = (x, y, z, ct)$

Spacetime interval:

$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$

Metric:

$g_{\mu\nu} = \mathrm{diag}(1, 1, 1, -1)$

Four-dimensional derivative:

$\partial_\mu = \partial/\partial x^\mu$

Wave operator:

$\Box = g^{\mu\nu}\partial_\mu\partial_\nu$

Explicit wave operator:

$\Box = \nabla^2 - (1/c^2)\partial^2/\partial t^2$

Massless scalar-field equation:

$\Box\Phi = 0$

Expanded equation:

$\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2 - \partial^2\Phi/\partial(ct)^2 = 0$

Equivalent classical form:

$\partial^2\Phi/\partial t^2 = c^2\nabla^2\Phi$

Plane wave:

$\Phi = A\, e^{i(k\cdot r - \omega t)}$

Dispersion relation:

$\omega^2 = c^2k^2$

Four-wave-vector:

$k^\mu = (k_x, k_y, k_z, \omega/c)$

Null condition:

$k_\mu k^\mu = 0$

Massless limit:

$m = 0 \rightarrow \Box\Phi = 0$

Central proposition:

A free massless classical scalar field propagating on Minkowski spacetime satisfies the Lorentz-invariant equation

$\Box\Phi = 0$

This preserves the original paper's structure and content while changing the scalar-field notation from φ to capital Φ.

2nd Research Paper

THE CLASSICAL SCALAR FIELD PERTURBATION EQUATION: A LORENTZ-INVARIANT FORMULATION FOR A PROPOSED GRAVITATIONAL SCALAR WAVE MODEL

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Abstract

This paper presents a perturbative formulation of a classical massless scalar field using the Lorentz-invariant wave equation in four-dimensional Minkowski spacetime. The total scalar field is separated into a background field and a small perturbation. The formulation is expressed using the following relation:

$\Phi = \Phi_0 + \delta\Phi$

Here, Φ represents the total scalar field, Φ_0 represents the background scalar field, and $\delta\Phi$ represents the scalar-field perturbation.

Starting from the massless scalar wave equation, $\Box\Phi = 0$, the paper derives the corresponding equation for the perturbation under the assumption that the background field independently satisfies the same wave equation. The resulting perturbation equation describes the propagation of a free massless scalar disturbance at the speed of light in flat spacetime.

This formulation provides a mathematical starting point for investigating a proposed gravitational scalar wave model. However, the scalar wave equation alone does not establish the existence of a physical gravitational scalar mode. Such a claim requires a defined gravitational theory, field equations, coupling to matter, and an analysis of the physical degrees of freedom.

Keywords: Classical scalar field, scalar-field perturbation, Lorentz invariance, massless wave equation, Minkowski spacetime, gravitational scalar wave model.

1. Introduction

Wave equations describe the propagation of disturbances through space and time. The classical scalar wave equation is one of the fundamental mathematical descriptions of wave propagation. It applies to scalar quantities whose evolution can be described by a second-order differential equation.

A scalar field assigns a scalar value to each point in spacetime. In a relativistic formulation, the scalar field can be represented by Φ and its propagation can be described using the Lorentz-invariant wave operator.

The massless scalar wave equation is:

$$\Box\Phi = 0$$

In this equation, □ represents the d'Alembert wave operator. In flat Minkowski spacetime, using the sign convention adopted in this paper, it is defined as:

$$\Box = \nabla^2 - (1/c^2)\,\partial^2/\partial t^2$$

Here, ∇^2 is the spatial Laplacian, c is the speed of light in vacuum, and t represents time.

The purpose of this paper is to develop a perturbative form of this equation. Instead of treating the total field as a single quantity, the field is separated into a background component and a small disturbance. This approach makes it possible to study the mathematical propagation of small changes in the scalar field.

The paper uses the term “proposed gravitational scalar wave model” to describe a possible direction for further investigation. It does not assume that every classical scalar wave is a gravitational wave or that the equation presented here independently establishes a new gravitational degree of freedom.

2. The Classical Massless Scalar Wave Equation

The starting point is the Lorentz-invariant equation for a free massless scalar field:

$$\Box\Phi = 0$$

Expanding the wave operator gives:

$$\nabla^2\Phi - (1/c^2)\,\partial^2\Phi/\partial t^2 = 0$$

Rearranging the equation produces:

$$\partial^2\Phi/\partial t^2 = c^2\nabla^2\Phi$$

This equation describes the propagation of a free massless scalar field in flat spacetime.

The equation is called Lorentz-invariant because it is expressed using the relativistic wave operator. Its form is preserved under Lorentz transformations when the field and coordinates are transformed consistently.

The equation is a mathematical model for a scalar field. Whether the field represents a physical gravitational quantity depends on additional theoretical definitions and equations.

3. Decomposition of the Total Scalar Field

To investigate small disturbances, the total scalar field is separated into two components:

$\Phi = \Phi_0 + \delta\Phi$

The symbols have the following meanings:

Φ: The total scalar field.

Φ_0: The background scalar field.

$\delta\Phi$: The perturbation, representing a small change relative to the background.

This notation distinguishes the background from the disturbance without using a prime symbol. The prime notation can have other meanings in mathematics, including differentiation, so $\delta\Phi$ provides a clearer designation for the perturbation.

The decomposition is a standard mathematical method for studying small disturbances around a specified background.

The decomposition itself does not require the perturbation to be small mathematically. The approximation becomes important when nonlinear equations are expanded and terms involving products or higher powers of the perturbation are neglected.

4. Derivation of the Scalar-Field Perturbation Equation

The original field equation is:

$\Box\Phi = 0$

Substitute the decomposition:

$\Phi = \Phi_0 + \delta\Phi$

The equation becomes:

$\Box(\Phi_0 + \delta\Phi) = 0$

Because the wave operator is linear, it acts separately on each component:

$\Box\Phi_0 + \Box\delta\Phi = 0$

This is the general result obtained from the decomposition.

Now assume that the background field satisfies the free massless scalar wave equation:

$\Box\Phi_0 = 0$

Substituting this condition gives:

$0 + \Box\delta\Phi = 0$

Therefore, the perturbation satisfies:

$$\boldsymbol{\Box\delta\Phi = 0}$$

This is the scalar-field perturbation equation derived under the stated background assumption.

The result follows directly from the linearity of the massless scalar wave operator. It shows that, when both the total field and its background satisfy the same homogeneous equation, the perturbation also satisfies that equation.

If the background does not satisfy $\Box\Phi_0 = 0$, the perturbation equation is instead:

$\Box\delta\Phi = -\Box\Phi_0$

Consequently, the homogeneous perturbation equation depends on the background condition and should not be assumed in every physical situation.

5. Expanded Form of the Perturbation Equation

Using the flat-spacetime wave operator,

$\Box = \nabla^2 - (1/c^2)\,\partial^2/\partial t^2$

the perturbation equation becomes:

$\nabla^2\delta\Phi - (1/c^2)\,\partial^2\delta\Phi/\partial t^2 = 0$

Rearranging gives:

$$\boldsymbol{\partial^2\delta\Phi/\partial t^2 = c^2\nabla^2\delta\Phi}$$

This equation describes the propagation of a free massless scalar disturbance in flat spacetime.

The equation has the same mathematical form as the original equation for the total scalar field. This is expected because the governing equation is linear and the background satisfies the same homogeneous equation.

The perturbation equation does not introduce a new propagation law by itself. Instead, it expresses the original wave equation in terms of the disturbance relative to the background.

6. Plane-Wave Solution

A plane-wave solution for the perturbation may be written in complex notation as:

$$\delta\Phi(x, t) = A \exp[i(k \cdot x - \omega t)]$$

Here:

A is the wave amplitude.

k is the wave vector.

x is the spatial position vector.

ω is the angular frequency.

t is time.

i is the imaginary unit.

Substitution into the massless scalar wave equation yields the dispersion relation:

$$\omega^2 = c^2|k|^2$$

For a wave with positive angular frequency and nonzero wave number, this gives:

$$\omega = c|k|$$

The phase speed is therefore:

$$v = \omega/|k| = c$$

This result describes the propagation speed of the plane-wave solution for a free massless scalar field in flat spacetime.

It does not, on its own, prove that a physical gravitational scalar wave exists or that such a wave has been experimentally observed.

7. Lorentz-Invariant Interpretation

The perturbation equation,

$$\Box \delta\Phi = 0$$

uses the same Lorentz-invariant wave operator as the original field equation.

If $\delta\Phi$ is a scalar field, its value at a spacetime event is invariant under a change of inertial coordinates, while its coordinate representation is evaluated using the transformed coordinates. The wave equation retains its form under Lorentz transformations.

This property makes the equation suitable for describing a free massless scalar disturbance within special relativity.

However, Lorentz invariance is a mathematical property of the equation and is not sufficient to establish the physical identity of the field. A field proposed to represent a gravitational degree of freedom must also be defined within an appropriate gravitational theory.

8. Connection to a Proposed Gravitational Scalar Wave Model

The present formulation may serve as a starting point for investigating whether a scalar disturbance could arise in a gravitational model.

For that purpose, it is necessary to distinguish two claims.

The first claim is mathematical: a scalar field satisfying the free massless wave equation admits perturbations that satisfy the same homogeneous equation when the background also satisfies it. This follows from the derivation presented above.

The second claim is physical: a scalar perturbation represents a gravitational wave or a gravitational degree of freedom. This claim requires additional theoretical support.

In general relativity, gravitational waves are described by perturbations of the spacetime metric and obey equations derived from the gravitational field equations. The free scalar equation presented here is not, by itself, a derivation of a gravitational wave equation.

To establish a gravitational interpretation for $\delta\Phi$, a future formulation would need to specify at least the following:

1. The physical definition of the scalar field Φ.
2. The gravitational field equations from which the scalar equation follows.
3. Whether the scalar field is an independent field or a quantity constructed from the metric.
4. How the field couples to matter and energy.
5. Whether the perturbation represents a physical degree of freedom rather than a mathematical reparameterization.

6. The relevant initial conditions, boundary conditions, and conservation laws.
7. The observational consequences that could distinguish the proposed model from established gravitational theories.

Without these additional elements, the equation should be interpreted as a classical scalar-field perturbation equation rather than a fully established gravitational scalar wave equation.

9. Extension to Curved Spacetime

The flat-spacetime formulation uses the operator □ defined in terms of ordinary spatial and temporal derivatives.

In curved spacetime, the corresponding scalar wave operator is generally written as:

$$\Box_g \Phi = 0$$

Here, $\Box_g$ denotes the covariant d'Alembert operator associated with the spacetime metric.

For a scalar field, it may be expressed in terms of the metric determinant and covariant metric components. Its detailed form depends on the chosen metric and coordinate system.

A curved-spacetime perturbation analysis must account for the background geometry as well as the background scalar field. If the metric is itself perturbed, the wave operator can also change under perturbation. In that situation, the equation for the scalar disturbance may contain terms involving the metric perturbation and its interaction with the background field.

Therefore, the flat-spacetime result,

$$\Box \delta\Phi = 0$$

cannot automatically be transferred to a general curved spacetime or to a complete gravitational theory. Such an extension requires a consistent perturbation of the full field equations.

10. Limitations of the Present Formulation

The present paper establishes a mathematical result within a specified model: if the total scalar field satisfies the homogeneous massless wave equation and the background field independently satisfies that equation, the perturbation also satisfies the homogeneous massless wave equation.

Several limitations must be acknowledged.

First, the equation does not identify the physical origin of the scalar field.

Second, it does not derive the scalar equation from Einstein's field equations or from a specified alternative theory of gravity.

Third, it does not demonstrate that the scalar perturbation carries gravitational energy or produces a measurable gravitational effect.

Fourth, the plane-wave solution establishes the dispersion relation for the assumed free massless scalar model, not the existence of a new gravitational wave mode.

Finally, the equation has not been connected here to a complete set of experimental predictions. Such predictions would be necessary to evaluate the physical viability of a proposed gravitational scalar wave theory.

These limitations do not invalidate the mathematical derivation. They define the distinction between a valid scalar-field wave model and a fully developed physical theory of gravitational scalar waves.

11. Conclusion

This paper has developed a perturbative formulation of the classical massless scalar wave equation.

The total scalar field is expressed as:

$\Phi = \Phi_0 + \delta\Phi$

Starting from the original equation,

$\Box\Phi = 0$

the decomposition gives:

$\Box\Phi_0 + \Box\delta\Phi = 0$

When the background field satisfies,

$\Box\Phi_0 = 0$

the perturbation equation becomes:

$$\mathbf{\Box\delta\Phi = 0}$$

In expanded form, this is:

$$\mathbf{\partial^2\delta\Phi/\partial t^2 = c^2\nabla^2\delta\Phi}$$

A plane-wave solution leads to the dispersion relation:

$\omega^2 = c^2|k|^2$

Thus, within the assumptions of the model, the scalar perturbation propagates according to the free massless scalar wave equation in flat spacetime.

The principal contribution of this paper is the explicit separation of the total scalar field into a background and a perturbation, together with the derivation of the resulting perturbation equation.

The formulation can be used as a mathematical starting point for further investigation into a proposed gravitational scalar wave model. However, the existence and physical interpretation of a gravitational scalar mode must be established through a defined gravitational theory, consistent field equations, and testable predictions.

References

Carroll, S. M. (2004). *Spacetime and Geometry: An Introduction to General Relativity*. Addison-Wesley.

Gordon, W. (1926). Der Comptonsche Effekt nach der Schrödingerschen Theorie. *Zeitschrift für Physik*, 40, 117–133.

Klein, O. (1926). Quantentheorie und fünfdimensionale Relativitätstheorie. *Zeitschrift für Physik*, 37, 895–906.

Wald, R. M. (1984). *General Relativity*. University of Chicago Press.

Author's Note

This paper presents a mathematical formulation of a classical scalar-field perturbation equation. The interpretation of the perturbation as a gravitational scalar wave is proposed as a direction for further theoretical research, not presented as an established result of general relativity or as an experimentally confirmed discovery.